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A numerical solution of the stochastic discrete algebraic Riccati equation

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Abstract

This paper proposes two algorithms for solving a stochastic discrete algebraic Riccati equation which arises in a stochastic optimal control problem for the discrete-time system. Our algorithms are generalized version of Hower's algorithm. Algorithm I has the quadratic convergence but requires to solve a sequence of extended Lyapunov equation. On the other hand, Algorithm II only needs solutions of standard Lyapunov equations which can be solved easily but it has a linear convergence. By a numerical example, we shall show that Algorithm I is superior to Algorithm II in the case of large dimension.

1 Introduction

The SDARE (Stochastic Discrete Algebraic Riccati Equation) arises in many problems, e.g. stochastic optimal control, guaranteed cost control[1]

$$P = \bar{\Omega}^{\mathrm{T}}(P)P\bar{\Omega}(P) + A_0^{\mathrm{T}}\Omega^{\mathrm{T}}(P)R\Omega(P)A_0 + C^{\mathrm{T}}C + \Upsilon(P) \quad (1)$$

where P is desired symmetric positive semi-definite matrix and $A_i (i = 0, 1, \ldots, p), B, C, R$ are given matrices with appropriate size. Matrix functions $\Upsilon(P)$, $\Omega(P)$, $\bar{\Omega}(P)$ are defined as

$$\Upsilon(P) = \sum_{i=1}^{p} A_i^{\mathrm{T}} P A_i \quad (2)$$

$$\Omega(P) = (B^{\mathrm{T}}PB + R)^{-1}B^{\mathrm{T}}P \quad (3)$$

$$\bar{\Omega}(P) = (I - B\Omega(P))A_0 \quad (4)$$

where R is assumed to be positive definite. In the following text, we will give shorthand SDARE for (1).

It is well known for the standard continuous algebraic Riccati equation, Kleinman's algorithm[2] is effective which iteratively solves standard Lyapunov equation. This method has quadratic convergence under the assumptions that the system is controllable and stabilizable. Kono et al.[3] proposed an extended Kleinman's algorithm for the stochastic continuous algebraic Riccati equation, and considered the computational effort. The numerical example showed that the solution which iteratively solves the generalized Lyapunov equations is superior to the solution which iteratively solves the standard continuous algebraic Lyapunov equations. But, in this result, quadratic convergence of the algorithm has not been proven. On the other hand, for the standard discrete algebraic Riccati equation, Hewer[4] proposed a solution similar to Kleinman's method. He had shown that if the closed-loop system is asymptotically stable then the algorithm has quadratic convergence. After that, Guo[5] had showed if the eigenvalues of the closed-loop system is on the unit circle, then first term becomes superior.

2 Preliminary

For the existence of solution for SDARE, Tang et al.[1] had shown following theorem.

Theorem 1 If (A_0, B) is controllable, (C, A_0) is detectable and

$$\inf_F \| \sum_{j=0}^{\infty} ((A_0 - BF)^{\mathrm{T}})^j \Upsilon(I)(A_0 - BF)^j \| < 1 \quad (5)$$

is satisfied, then there exists positive semi-definite solution P of (1). Where $\|A\| = \max\{\sqrt{\lambda} : \lambda$ is a eigenvalue of $A^{\mathrm{T}}A\}$.

Next, under the same assumption in **theorem 1**, we consider extended DALE(Discrete Algebraic Lyapunov Equation)

$$V_0 = \bar{\Omega}_0^{\mathrm{T}} V_0 \bar{\Omega}_0 + A_0^{\mathrm{T}}\Omega_0^{\mathrm{T}} R \Omega_0 A_0 + C^{\mathrm{T}}C + \Upsilon(V_0) \quad (6)$$

where

$$\bar{\Omega}_0 = (I - B\Omega_0)A_0 \quad (7)$$

From **lemma 2** in [1], we have following proposition.

Proposition 1 There exists positive semi-definite solution V_0 of (6) if $\bar{\Omega}_0$ is asymptotically stable and

$$|| \sum_{j=0}^{\infty} ((A_0 - B\Omega_0 A_0)^{\mathrm{T}})^j \Upsilon(I)(A_0 - B\Omega_0 A_0)^j || < 1 \quad (8)$$

is satisfied.

When A_0 is $n \times n$ matrix, by using solution in [6] it needs n^6 ordered flops to solve equation (8). Thus, the computational effort becomes very large.

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3 Algorithm I

From the proof of **theorem 1** in [1], we have following theorem.

Theorem 2 Under the same assumption of **theorem 1**, let us consider the sequence of the extended DALE as

$$V_k = \bar{\Omega}_k^{\mathrm{T}} V_k \bar{\Omega}_k + A_0^{\mathrm{T}} \Omega_k^{\mathrm{T}} R \Omega_k A_0 + C^{\mathrm{T}} C + \Upsilon(V_k) \quad (9)$$

$$\Omega_k = (B^{\mathrm{T}} V_{k-1} B + R)^{-1} B^{\mathrm{T}} V_{k-1} \quad (10)$$

$$\bar{\Omega}_k = (I - B\Omega_k) A_0 \quad (11)$$

Then, we have the solution P of (1)

$$P = \lim_{k \to \infty} V_k \quad (12)$$

In this paper, we call the procedure in **theorem 1** **Algorithm I**. Following lemma which concerns to the existence of the solution of (1) is established.

Lemma 1 Let $\bar{\sigma}_k = \|\bar{\Omega}_k\|$. If for the closed loop system which is constructed by using solution of (1),

$$\bar{\sigma} = \|\bar{\Omega}\| < 1 \quad (13)$$

is satisfied, then there exists positive real number $\alpha_1 < 1$ which is independent of k and

$$\bar{\sigma}_k < \alpha_1 \quad (14)$$

Lemma 2 There exists α_2 which is independent of k and following inequality is satisfied.

$$\|A_0^{\mathrm{T}}(\Omega - \Omega_{k+1})^{\mathrm{T}}(B^{\mathrm{T}} P B + R)(\Omega - \Omega_{k+1}) A_0\| \leq \alpha_2 \|P - V_k\|^2 \quad (15)$$

Lemma 3 For the solution of (1), following equality is satisfied.

$$V_{k+1} - P = \sum_{j=0}^{\infty} (\bar{\Omega}_{k+1}^{\mathrm{T}})^j \{\Upsilon(V_{k+1}) - \Upsilon(P) + A_0^{\mathrm{T}}(\Omega - \Omega_{k+1})^{\mathrm{T}} \cdot (B^{\mathrm{T}} P B + R)(\Omega - \Omega_{k+1}) A_0\} \bar{\Omega}_{k+1}^j \quad (16)$$

The next theorem is the main result of this paper which guarantees quadratic convergence of the **Algorithm I** under additional assumptions.

Theorem 3 Let us assume (13) is satisfied for the closed-loop system which is obtained by using positive definite solution P of (1) and $\|\Upsilon(I)\| + \alpha_1^2 < 1$. Then the convergence of **Algorithm I** is

$$\|P - V_{k+1}\| \leq c \|P - V_k\|^2 \quad (17)$$

where

$$c = \alpha_2 (1 - \alpha_1^2 - \|\Upsilon(I)\|) \quad (18)$$

(Proof) From **lemma 3**,

$$\|V_{k+1} - P\| \leq \|\sum_{j=0}^{\infty} (\bar{\Omega}_{k+1}^{\mathrm{T}})^j \{\Upsilon(V_{k+1}) - \Upsilon(P)\} \bar{\Omega}_{k+1}^j\| + \|\sum_{j=0}^{\infty} (\bar{\Omega}_{k+1}^{\mathrm{T}})^j A_0^{\mathrm{T}} (\Omega - \Omega_{k+1})^{\mathrm{T}} (B^T P B + R) \cdot (\Omega - \Omega_{k+1}) A_0 \bar{\Omega}_{k+1}^j\| \quad (19)$$

Since $\bar{\sigma} < 1$, from **lemma 1** the first term becomes

$$\begin{aligned} &\|\sum_{j=0}^{\infty} (\bar{\Omega}_{k+1}^{\mathrm{T}}) \{\Upsilon(V_{k+1}) - \Upsilon(P)\} \bar{\Omega}_{k+1}^j\| \\ &\leq \sum_{j=0}^{\infty} \|(\bar{\Omega}_{k+1}^{\mathrm{T}})^j\| \cdot \|\Upsilon(V_{k+1} - P)\| \cdot \|\bar{\Omega}_{k+1}^j\| \\ &= \sum_{j=0}^{\infty} \bar{\sigma}_{k+1}^{2j} \|\Upsilon(V_{k+1} - P)\| \\ &= \frac{\|\Upsilon(V_{k+1} - P)\|}{1 - \bar{\sigma}_{k+1}^2} \leq \frac{\|\Upsilon(V_{k+1} - P)\|}{1 - \alpha_1^2} \\ &\leq \frac{\|V_{k+1} - P\| \cdot \|\Upsilon(I)\|}{1 - \alpha_1^2} \end{aligned} \quad (20)$$

Let

$$\alpha_3 = \frac{\|\Upsilon(I)\|}{1 - \alpha_1^2}$$

then from the assumption of theorem, $\alpha_3 < 1$ and

$$\|\sum_{j=0}^{\infty} (\bar{\Omega}_{k+1}^{\mathrm{T}})^j \{\Upsilon(V_{k+1}) - \Upsilon(P)\} \bar{\Omega}_{k+1}^j\| \leq \alpha_3 \|V_{k+1} - P\| \quad (21)$$

is satisfied. Next, note that the second term in (19) and from **lemma 1**

$$\begin{aligned} &\|\sum_{j=0}^{\infty} (\bar{\Omega}_{k+1}^{\mathrm{T}})^j A_0^{\mathrm{T}} (\Omega - \Omega_{k+1})^{\mathrm{T}} \cdot (B^{\mathrm{T}} P B + R)(\Omega - \Omega_{k+1}) A_0 \bar{\Omega}_{k+1}^j\| \\ &\leq \|A_0^{\mathrm{T}} (\Omega - \Omega_{k+1})^{\mathrm{T}} (B^{\mathrm{T}} P B + R)(\Omega - \Omega_{k+1}) A_0\| / (1 - \alpha_1^2) \end{aligned} \quad (22)$$

Thus

$$\|V_{k+1} - P\| \leq \alpha_3 \|V_{k+1} - P\| + \|A_0^{\mathrm{T}} (\Omega - \Omega_{k+1})^{\mathrm{T}} (B^{\mathrm{T}} P B + R)(\Omega - \Omega_{k+1}) A_0\| / (1 - \alpha_1^2) \quad (23)$$

and we obtain

$$\|V_{k+1} - P\| \leq \|A_0^{\mathrm{T}} (\Omega - \Omega_{k+1})^{\mathrm{T}} (B^{\mathrm{T}} P B + R)(\Omega - \Omega_{k+1}) A_0\| / (1 - \alpha_1^2)(1 - \alpha_3) \quad (24)$$

Consequently, we obtain (17) from **lemma 2**.

□

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4 Algorithm II

In this section, we shall show the **Algorithm II** which is extended solution for the stochastic continuous algebraic Riccati equation [3] to the discrete case.

Theorem 4 Under the same assumption of **theorem 1**, let us consider the following DALE sequence

$$V_k = \bar{\Omega}_k^{\mathrm{T}} V_k \bar{\Omega}_k + A_0^{\mathrm{T}} \Omega_k^{\mathrm{T}} R \Omega_k A_0 + C^{\mathrm{T}} C + \Upsilon_k, \quad k = 1, 2, 3, \ldots \tag{25}$$

where

$$\Upsilon_k = \Upsilon(V_{k-1}) = \sum_{i=1}^{p} A_i^{\mathrm{T}} V_{k-1} A_i \tag{26}$$

$$\Omega_k = (B^{\mathrm{T}} V_{k-1} B + R)^{-1} B^{\mathrm{T}} V_{k-1} \tag{27}$$

$$\bar{\Omega}_k = (I - B\Omega_k) A_0 \tag{28}$$

Then we obtain the solution P of (1) as

$$P = \lim_{k \to \infty} V_k \tag{29}$$

When the system size is $n \times n$, it needs n^4 ordered flops to solve (25) by Bartels-Stewart method. Then, the computational effort is less than the extended DALE.

Before the proof of **theorem 4**, we shall show the following lemma.

Lemma 4 Let us assume there exist appropriate size matrices A_0, B, C and positive definite symmetric matrices R, V. For arbitrary Ω_0 and Ω_1 where

$$\Omega_1 = (B^{\mathrm{T}} V B + R)^{-1} B^{\mathrm{T}} V \tag{30}$$

Then, following identical equation is satisfied

$$\begin{aligned} & A_0^{\mathrm{T}} (\Omega_1 - \Omega_0)^{\mathrm{T}} (B^{\mathrm{T}} V B + R)(\Omega_1 - \Omega_0) A_0 \\ & + \bar{\Omega}_1^{\mathrm{T}} V \bar{\Omega}_1 + A_0^{\mathrm{T}} \Omega_1^{\mathrm{T}} R \Omega_1 A_0 \\ & = \bar{\Omega}_0^{\mathrm{T}} V \bar{\Omega}_0 + A_0^{\mathrm{T}} \Omega_0^{\mathrm{T}} R \Omega_0 A_0 \end{aligned} \tag{31}$$

where $\bar{\Omega}_0$ is defined by (7) and $\bar{\Omega}_1 = (I - B\Omega_1) A_0$.

(Proof) From (6) and **lemma 4**,

$$\begin{aligned} V_0 &= \bar{\Omega}^{\mathrm{T}} V_0 \bar{\Omega}_0 + A_0^{\mathrm{T}} \Omega_0^{\mathrm{T}} R \Omega_0 A_0 + C^{\mathrm{T}} C + \Upsilon(V_0) \\ &= \bar{\Omega}_1^{\mathrm{T}} V_0 \bar{\Omega}_1 + A_0^{\mathrm{T}} \Omega_1^{\mathrm{T}} R \Omega_1 A_0 + C^{\mathrm{T}} C + \Upsilon(V_0) \\ &\quad + A_0^{\mathrm{T}} (\Omega_1 - \Omega_0)^{\mathrm{T}} (B^{\mathrm{T}} V_0 B + R)(\Omega_1 - \Omega_0) A_0 \\ &= \bar{\Omega}_1^{\mathrm{T}} V_0 \bar{\Omega}_1 + M_1 \end{aligned} \tag{32}$$

where

$$\begin{aligned} M_1 &= A_0^{\mathrm{T}} \Omega_1^{\mathrm{T}} R \Omega_1 A_0 + C^{\mathrm{T}} C + \Upsilon(V_0) \\ &\quad + A_0^{\mathrm{T}} (\Omega_1 - \Omega_0)^{\mathrm{T}} (B^{\mathrm{T}} V_0 B + R)(\Omega_1 - \Omega_0) A_0 \end{aligned} \tag{33}$$

From **lemma 2 ii)** of [1], $\bar{\Omega}_1$ is asymptotically stable, (32) is equivalent to

$$V_0 = \sum_{j=0}^{\infty} (\bar{\Omega}_1^{\mathrm{T}})^j M_1 \bar{\Omega}_1^j \tag{34}$$

On the other hand, the solution V_1 where $k = 1$ in (25) is

$$V_1 = \sum_{j=0}^{\infty} (\bar{\Omega}_1^{\mathrm{T}})^j (A_0^{\mathrm{T}} \Omega_1^{\mathrm{T}} R \Omega_1 A_0 + C^{\mathrm{T}} C + \Upsilon_1) \bar{\Omega}_1^j \tag{35}$$

Note that $\Upsilon_1 = \Upsilon(V_0)$, then

$$\begin{aligned} V_1 - V_0 = & -\sum_{j=0}^{\infty} (\bar{\Omega}_1^{\mathrm{T}})^j A_0^{\mathrm{T}} (\Omega_1 - \Omega_0)^{\mathrm{T}} \\ & \cdot (B^{\mathrm{T}} V_0 B + R)(\Omega_1 - \Omega_0) A_0 \bar{\Omega}_1^j \\ \leq & \ 0 \end{aligned} \tag{36}$$

In general, let us assume V_{k-1} is positive definite and

$$V_k - V_{k-1} \leq 0 \tag{37}$$

Then, from (25) and **lemma 4**,

$$\begin{aligned} V_k &= \bar{\Omega}_k^{\mathrm{T}} V_k \bar{\Omega}_k + A_0^{\mathrm{T}} \Omega_k^{\mathrm{T}} R \Omega_k A_0 + C^{\mathrm{T}} C + \Upsilon_k \\ &= \bar{\Omega}_{k+1}^{\mathrm{T}} V_k \bar{\Omega}_{k+1} + A_0^{\mathrm{T}} \Omega_{k+1}^{\mathrm{T}} R \Omega_{k+1} A_0 + C^{\mathrm{T}} C + \Upsilon_k \\ &\quad + A_0^{\mathrm{T}} (\Omega_{k+1} - \Omega_k^{\mathrm{T}}) (B^{\mathrm{T}} V_k B + R)(\Omega_{k+1} - \Omega_k) A_0 \\ &= \bar{\Omega}_{k+1}^{\mathrm{T}} V_k \bar{\Omega}_{k+1} + M_{k+1} \end{aligned} \tag{38}$$

where

$$\begin{aligned} M_{k+1} &= A_0^{\mathrm{T}} \Omega_{k+1}^{\mathrm{T}} R \Omega_{k+1} A_0 + C^{\mathrm{T}} C + \Upsilon_k \\ &\quad + A_0^{\mathrm{T}} (\Omega_{k+1} - \Omega_k)^{\mathrm{T}} (B^{\mathrm{T}} V_k B + R) \\ &\quad \cdot (\Omega_{k+1} - \Omega_k) A_0 \end{aligned} \tag{39}$$

As shown in above, $\bar{\Omega}_{k+1}$ is asymptotically stable, then positive semi-definite solution V_k of (25) is expressed as

$$V_k = \sum_{j=0}^{\infty} (\bar{\Omega}_{k+1}^{\mathrm{T}})^j M_{k+1} \bar{\Omega}_{k+1}^j \tag{40}$$

Since V_{k+1} is expressed as

$$\begin{aligned} V_{k+1} = & \sum_{j=0}^{\infty} (\bar{\Omega}_{k+1}^{\mathrm{T}})^j (A_0^{\mathrm{T}} \Omega_{k+1}^{\mathrm{T}} R \Omega_{k+1} A_0 \\ & + C^{\mathrm{T}} C + \Upsilon_{k+1}) \bar{\Omega}_{k+1}^j \end{aligned} \tag{41}$$

Then we obtain

$$\begin{aligned} & V_{k+1} - V_k \\ & = \sum_{j=0}^{\infty} (\bar{\Omega}_{k+1}^{\mathrm{T}})^j \{ \Upsilon_{k+1} - \Upsilon_k - A_0^{\mathrm{T}} (\Omega_{k+1} - \Omega_k)^{\mathrm{T}} \\ & \quad \cdot (B^{\mathrm{T}} V_k B + R)(\Omega_{k+1} - \Omega_k) A_0 \} \bar{\Omega}_{k+1}^j \end{aligned} \tag{42}$$

From the assumption (37) and linearity of $\Upsilon(P)$,

$$\begin{aligned}\Upsilon_{k+1} - \Upsilon_k &= \Upsilon(V_k) - \Upsilon(V_{k-1}) \\ &\leq 0\end{aligned}$$

then by using (42), following inequality is satisfied.

$$V_{k+1} - V_k \leq 0$$

Since V_k is monotonically non-increase and bounded below

$$V_\infty = \lim_{k\to\infty} V_k$$

Cleary, V_∞ is solution of (1)

□

Theorem 5 Convergence of **Algorithm II** is represented as

$$\|P - V_{k+1}\| \leq c_1\|P - V_k\| + c_2\|P - V_k\|^2 \qquad (43)$$

where c_1 and c_2 are positive constant scalar which are independent of k.

(Proof) In the same way as proof of **lemma 3**,

$$\begin{aligned}&V_{k+1} - P \\ &= \sum_{j=0}^{\infty}(\bar{\Omega}_{k+1}^{\mathrm{T}})^j\{\Upsilon_{k+1} - \Upsilon(P) + A_0^{\mathrm{T}}(\Omega - \Omega_{k+1})^{\mathrm{T}} \\ &\quad \cdot (B^{\mathrm{T}}PB + R)(\Omega - \Omega_{k+1})A_0\}\bar{\Omega}_{k+1}^j \qquad (44)\end{aligned}$$

From **lemma 1**, there exists α_1 which is independent of k and the norm of both side of (44) is

$$\begin{aligned}&\|P - V_{k+1}\| \\ &\leq \sum_{j=0}^{\infty}\|\bar{\Omega}_{k+1}^j\|^2 \cdot \|\Upsilon_{k+1} - \Upsilon(P) \\ &\quad + A_0^{\mathrm{T}}(\Omega - \Omega_{k+1})^{\mathrm{T}}(B^{\mathrm{T}}PB + R)(\Omega - \Omega_{k+1})A_0\| \\ &\leq \{\|\Upsilon_{k+1} - \Upsilon(P)\| + \|A_0^{\mathrm{T}}(\Omega - \Omega_{k+1})^{\mathrm{T}} \\ &\quad \cdot (B^{\mathrm{T}}PB + R)(\Omega - \Omega_{k+1})A_0\|\}/(1 - \alpha_1^2) \qquad (45)\end{aligned}$$

From (45) and **lemma 2**, **theorem 5** has benn proven.

□

5 Numerical Example

In this section, we show the numerical example of $A_0, A_1 \in R^{n\times n}$, $B \in R^{n\times 1}$ and $C \in R^{1\times n}$ are random matrices and $p = 1$. Following table illustrates the results of $n = 5, 7, 9, 11, 13$. F_1 and I_1 are flops and iteration number for convergence of **Algorithm I**. F_2 and I_2 are result of **Algorithm II**.

Table 1 : Comparison of two methods

n	F_1/F_2	I_1/I_2
5	2.3222	0.6667
7	2.6266	0.4444
9	0.8723	0.0714
11	0.1648	0.0073
13	7.4298×10^{-5}	1.8716×10^{-8}

For $n = 1 \sim 7$, **Algorithm II** is more superior to **Algorithm I**, same as continuous time case. But, for $n = 7$, **Algorithm I** is more advantageous, and for more large number, this tendency becomes more remarkable. This result shows the quadratic convergence of **Algorithm I** for the higher order problem.

6 Conclusion

We proposed two algorithms for solution of SDARE and compare these computaional effort. Future study is consideration of convergence when the closed loop system has eigenvalues on the unit circle.

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